Revisiting Geometric Construction using Geogebra

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Abstract

Construction problems have always been an important part in learning Geometry. Mastering construction helps students in logical reasoning. In this paper, we will take a look at traditional construction problems and create these constructions using GeoGebra. GeoGebra, as a software, has many functions. However, in this paper, we will only make use of functions that mimics the traditional compass and straightedge construction.

We will start with simple construction such as constructing angles and triangles. We will discuss construction of angle bisectors. We also use construction in showing certain properties of geometric objects, such as triangles and circles. We look at properties of angle bisectors and side bisectors of triangles, as well as chords of a circle. Finally, we will build upon these basic construction techniques to eventually show and construct more complicated theorems.

1. Introduction

Geometric construction has always been a fascination to many mathematicians and educators. While restricting the tools to straight edge and compass is not practical for real life construction, studies show that the exercises help students think logically [12]. Furthermore, geometric construction reflects the axiomatic system of Euclidean geometry. There is a rich supply of construction problems that can be analyzed from various old and new sources. In analyzing why certain constructions work, the students will be able to visualize how certain properties and formulas work.

In solving the various construction problems, we will make use of the software GeoGebra [11]. Many recent papers on Geometric construction, such as [2, 13], make use of dynamic geometry software. In particular, GeoGebra came out in 2002 as a free dynamic geometry software, with comparable functionalities as other proprietary software. Currently GeoGebra is at version 4.4, with version 5 at the beta release.

Works such as [9, 10] have explored the effects of using GeoGebra in teaching various math lessons. Using dynamic geometry software has many advantages in classroom discussions. During lesson planning, teachers can already create the GeoGebra files to be used for class. With the prepared file, the teacher has extra time to create a more stimulating discussion in classes. Furthermore, the software is very handy as teachers react to student questions, comments and conjectures.

At our college, most of the undergraduate math classes we teach make use of some software (both mathematical and otherwise) to aid in student learning. In our various syllabi, we explicitly state that exercises in software such as Microsoft Excel and MatLab are done during laboratory classes to learn certain topics. It is our opinion that these software may not be as effective for students who are using a visual learning technique, and for topics that need to be taught visually. Slowly, some members of our faculty began exploring other software, especially those that are more helpful visually. One of the software we started using is Geogebra.

In this paper, we take a look at two complex construction problems: a Japanese sangaku problem involving four incircles inside an equilateral triangle, and the Archimedean shoemaker problem. It is worthwhile to mention that the solution to the shoemaker problem makes use of two special cases of the solution to the classical Problem of Apollonius.

2. An equilateral triangle with four congruent incircles

This first problem is a Sangaku construction problem. Sangakus are wooden tablets inscribed with problems in Euclidean geometry offered by the Japanese at Shinto shrines or Buddhist temples during the Japanese isolation period (1603-1867). Sangaku problems are diverse (they are not just construction problems!) and provide a rich material both for teaching mathematics and research. Today, several references [4, 5, 6, 15, 16, 19] discuss Sangaku problems extensively. In particular, the theorem below is Problem 2.1.7 in [4].

This Sangaku construction problem is interesting because students will make use of constructing midpoints of a line segment, perpendicular line, angle bisector, and incircle of a triangle. This construction problem can be summarized in the following theorem:

Theorem 2.1. Given an equilateral triangle of side a, a line through each vertex can be constructed so that the incircles of the four triangles formed are congruent. Furthermore, the incircles all have radii $\frac{1}{8}\left(\sqrt{7}-\sqrt{3}\right)a$.

The existence of the three suitable lines to form the congruent incircles can be shown through construction. Furthermore, when we use GeoGebra to construct, we can observe that changing the length of the side of the equilateral triangle will change the length of the radii by the multiplier $\frac{1}{8}\left(\sqrt{7}-\sqrt{3}\right)$. The method to obtain the multiplier is difficult to explain using Geogebra, and needs a separate discussion in the classroom (as opposed to using the computer laboratory for the constructions). The proof to the radii of the incircles is elementary but lengthy; three detailed proofs can be found in [1].

To start the construction, students may be asked to begin by constructing an equilateral triangle. We start by constructing the line segment AB. Next, we construct two circles: one whose center on A and through B while the other has center B through A. The two circles will have two points of intersection. We pick one and use it as the third vertex of our equilateral triangle ABC (see Figure 2.1.a).

The next step is to construct the three lines mentioned in Theorem 2.1. Our construction will focus on the line passing through vertex A, and the construction of the other two lines will be similar. This line is essentially a radius of a circle centered at A and passing through B (cf. line AL in Figure 3.1.b). In particular, this has to be the radius that intersects with a chord of the same circle; and this chord has to pass by the midpoint of BC and is perpendicular to AB (cf. line LK in Figure 3.1.b).

To begin the construction, we need to construct the midpoint of side BC. To do so, we construct the circles centered at B passing through C and centered at C passing through B. The two circles will have two intersections E and F. The intersection of line segment EF and side BC is the midpoint G of BC.

Next, we construct the line perpendicular to AB passing through G. Select G as the center of a circle passing through B. The intersection of this circle and the side AB is I. We then construct two circles: one centered at B passing through I and another centered at I passing through B. The intersection of these two new circles are G and K. We connect G and K to form the line perpendicular to AB passing through G.

We then go back to the earlier circle centered at A passing through B. We take the intersection of this earlier circle and the line GK to obtain point L. The line segment AL is the required line in Theorem 2.1 that passes through the vertex A (see Figure 2.1.b).

By a similar process, we can construct suitable lines passing through vertices B and C. Taking the intersection of these three lines and hiding the unnecessary circles and line segments, we form four triangles inside our original triangle ABC (see Figure 2.2.a).

The next step is to construct the incenters and incircles of the four interior triangles. We shall construct the incircle of triangle AOB and the process for the other three triangles are the same. The incenter is simply the intersection of the three angle bisectors of the interior angles of the triangle. To obtain the intersection, however, we only need to construct at least two of the three angle bisectors. We start with vertex A. Construct a circle centered at A passing through O. The intersection of this circle and the line segment AB is U. Construct two new circles, one centered at O passing through U and another centered at U passing through O. One of the intersections of the two new circles is W. Line segment AW bisects $\angle OAB$ (see Figure 2.2.b).

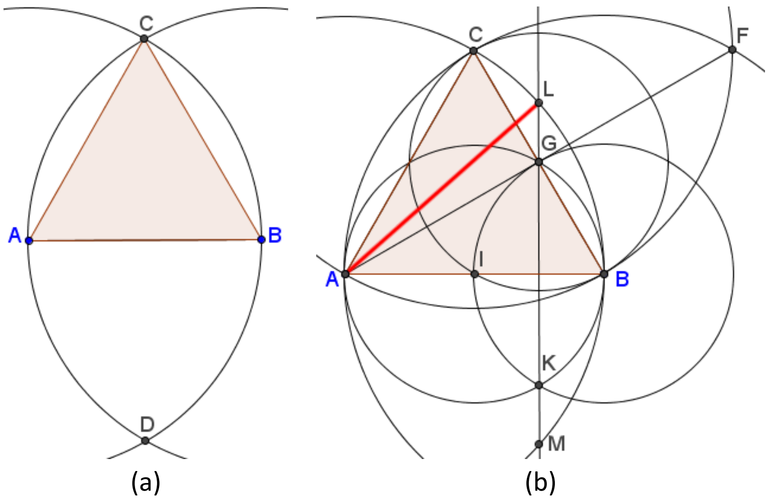

Figure 2.1: (a) *An equilateral triangle*; (b) *Constructing the suitable line from Theorem 2.1 passing through vertex A*

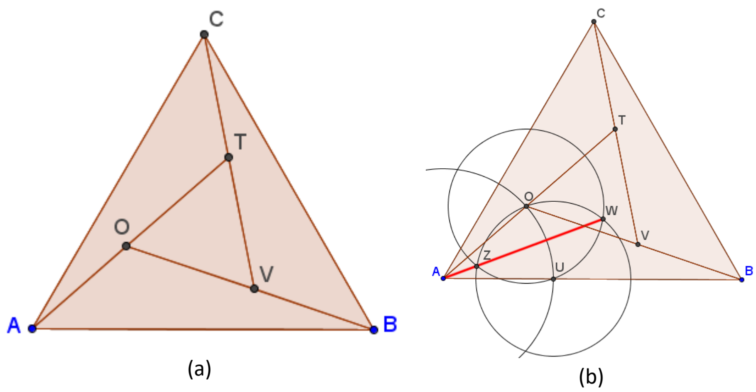

Figure 2.2: (a) *The equilateral triangle with the three lines from Theorem 2.1*; (b) *Constructing the angle bisector of* $\angle OAB$

We do a similar process for another angle, say $\angle ABO$. The intersection of the two angle bisectors is the incenter X of triangle AOB. Next, we construct a line segment passing through X and perpendicular to side AB. The intersection of AB and the perpendicular line passing through X is Y. Construct a circle centered at X passing through Y and this is the incircle of triangle AOB. We repeat the process for triangles ATC, BVC, and TOV.

Finally, we can use GeoGebra to check the measurements of the radii of the incircles as well as the measurement of side AB, which is a. According to Theorem 2.1, when $a = 1$, the radii of the incircles have measurement $\frac{1}{8}(\sqrt{7}-\sqrt{3}) \approx 0.11$ (see Figure 2.3.a). Also, when $a = 5$, the radii of the incircles have measurement $\frac{5}{8}(\sqrt{7}-\sqrt{3}) \approx 0.57$ (see Figure 2.3.b). The final GeoGebra file, Problem1.ggb, can be found in Section 6.

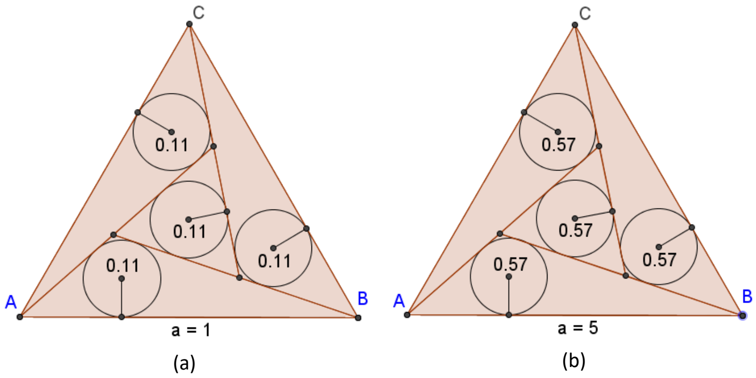

Figure 2.3: (a) *Checking Theorem 2.1 when* $a = 1$; (b) *Checking Theorem 2.1 when* $a = 5$

3. The Archimedean twin circles

The second problem we will discuss is interesting because it is an ancient problem. It was discussed in T.L. Heath's 1897 book *The Works of Archimedes* (Book of Lemmas, Proposition 5, p. 305 and Book of Lemmas, Proposition 6, p. 307 of [7]), as well as other references (p. 5 of [3], δ, p. 181 and δ, p. 416 of [8], Soddy's poem in [17], XI, p 325 of [18]). Consider the line segment AB with point P on AB. Suppose there are three circles with diameters AB, AP, and PB, where the radius of circle AP is a and the radius of circle PB is b. Let Q be the intersection of circle AB and the line perpendicular to AB passing through P (cf. Figure 3.1). Then we have the following results due to Archimedes:

Theorem 3.1. (a) We define the twin circles C_1 and C_2 as follows: C_1 is tangent to PQ, circle AB, and circle AP while C_2 is tangent to PQ, circle AB, and circle PB. Then C_1 and C_2 have equal radii and is given by

$$t = \frac{ab}{a+b}.$$

(b) The circle C tangent to circles AB, AP, and PB has radius

$$p = \frac{ab(a+b)}{a^2+ab+b^2}.$$

The theorem above is reminiscent of the classical problem of Apollonius, solved by Viète by construction in 1600 [18]. In the problem of Apollonius, we are asked to construct a circle that is tangent to three given circles. This problem led to several cases (in fact, 10 cases), depending on whether the given circles have zero, positive finite, or infinite radius. If a given circle has zero radius, then you are constructing a circle tangent to a point. If a given circle has infinite radius, then you are constructing a circle tangent to a line.

In Theorem 3.1.a, we are trying to construct a circle C_1 tangent to two circles and a line; or tangent to two circles with positive finite radius and a circle with infinite radius. The same is true in constructing C_2. The formula for the radii of circles C_1 and C_2 is a direct consequence of Proposition 5, p. 305 in [7]. In Theorem 3.1.b, we are trying to construct a circle C tangent to three circles of positive finite radius. The formula for the radius of circle C is a direct consequence of Proposition 6, p. 307 in [7].

Just like in the previous section, let us construct the figures described in the theorem and use examples to check if the formulas are true. We start by constructing the line segment AB and picking a point P in AB. Since AB, AP, and PB are diameters, we need to construct the midpoints C, D, and E so we can construct the circles AB, AP, and PB, respectively. By a similar method in the previous section, we also construct point Q by constructing the line perpendicular to AB passing through P (see Figure 3.1).

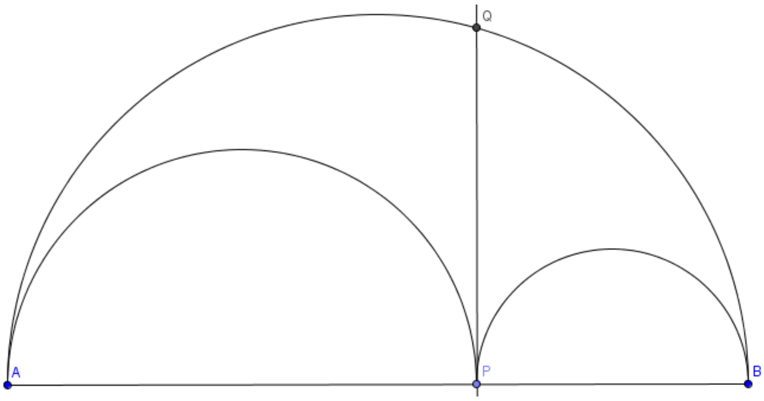

Figure 3.1: *Constructing circles* AB, AP, PB

The next step is to construct the twin circles C_1 and C_2. In this discussion, we construct circle C_1, and C_2 is constructed similarly. In the construction, we will see a small circle centered at P passing through a point H (see Figure 3.2). Circles C_1 and C_2 are basically this circle moved (or translated) upwards and horizontally. So the key is to construct this small circle, obtain its radius, and construct C_1 above it.

To construct the small circle centered at P, we construct the line segment FD, where FD is perpendicular to AB at D. Then we construct GE, where GE is perpendicular to AB at E. Then we find the intersection H of line segments DG and FE. Construct the circle centered at P passing through H. The intersection of this circle with AB are points I and J.

To construct C_1, we construct the circle centered at D passing through J and construct the line perpendicular to AB passing through I. The intersection L of the last circle and perpendicular line is the center of circle C_1. Next, construct the line perpendicular to PQ passing through L. The intersection M of this perpendicular line with PQ is the point of tangency of C_1 with PQ. So, C_1 is simply the circle centered at L passing through M (see Figure 3.2).

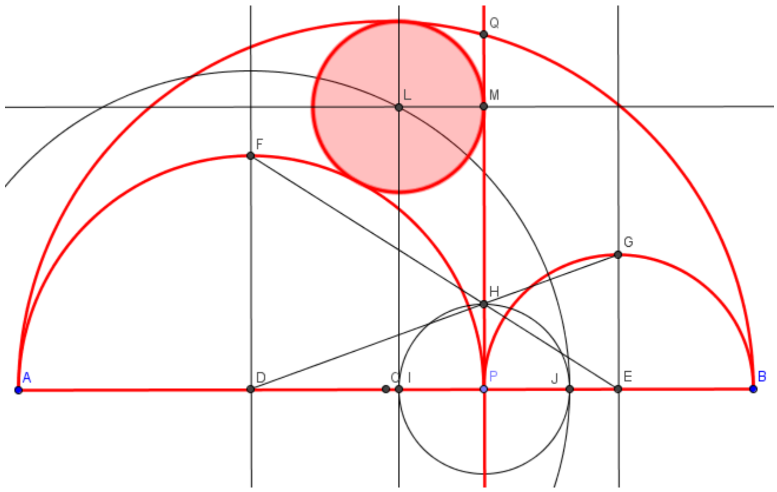

Figure 3.2: *Constructing circle* C_1

When C_2 has been constructed, we can now check Theorem 3.1.a. For example, when $a = 4$ and $b = 3$, $t = \frac{12}{7} \approx 1.71$ (see Figure 3.3). The final GeoGebra file, Problem2.ggb, can be found in Section 6.

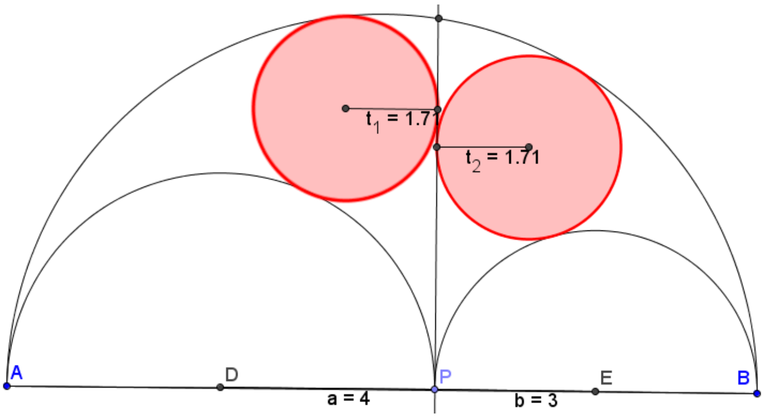

Figure 3.3: *Checking Theorem 3.1.a when $a = 4$ and $b = 3$*

Theorem 3.1.b is interesting for another reason. The construction involved is related to Soddy's circles [17]. The traditional statement of the problem in Soddy's circles is that given circle AB, three circles interior to circle AB can be constructed such that all four circles are mutually tangent to each other at a total of six points. In Theorem 3.1.b, however, the big circle AB and two of the three interior circles (circles AP and PB) are already given. The task is to construct the third circle C. In the end, the four circles will be mutually tangent at six points.

To construct circle C, we start at our three original circles: circles AB, AP, and PB, with centers C, D, and E, respectively. If in the previous construction, the key was the circle centered at P passing through a point H, here, the key is the circle centered at H passing through P.

To construct this small circle centered at H, we begin by constructing the line segment FD, where FD is perpendicular to AB at D. Then we construct GE, where GE is perpendicular to AB at E. Then we find the intersection H of line segments DG and FE. Now, construct the circle centered at H passing through P.

To construct the circle C, we find the intersection of circle AP with circle HP, which is I; we also find the intersection of circle PB with circle HP, which is J. The intersection of DI and EJ is L. Circle C is the circle centered at L passing through I and J (see Figure 3.4).

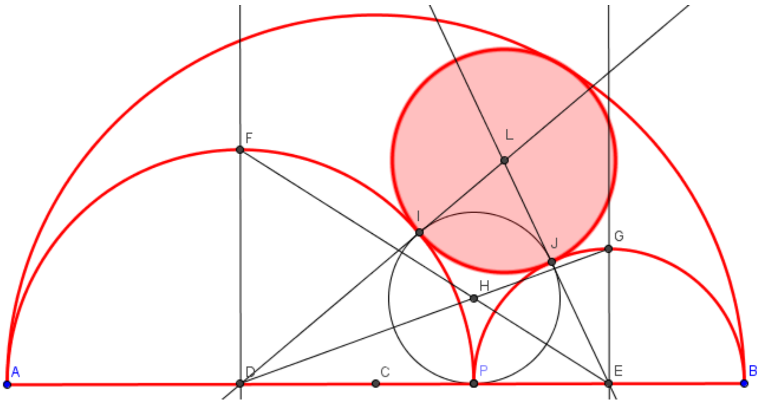

Figure 3.4: *Constructing circle C*

We can now check Theorem 3.1.b. For example, when $a = 3.7$ and $b = 2.2$, $p = \frac{ab(a+b)}{a^2+ab+b^2} = \frac{8.14\times5.9}{13.69+8.14+4.84} = \frac{48}{26.67} \approx 1.8$ (see Figure 3.5). The final GeoGebra file, Problem3.ggb, can be found in Section 6.

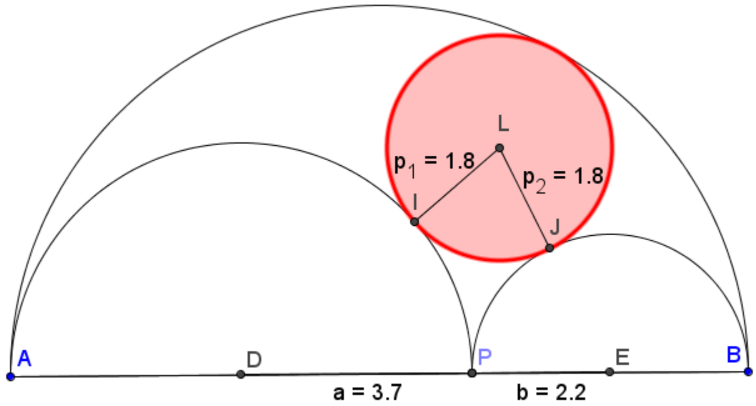

Figure 3.5: *Checking Theorem 3.1.b when $a = 3.7$ and $b = 2.2$*

4. Concluding Remarks

In this short note, we have seen solved construction problems using GeoGebra. While using GeoGebra for construction is a good idea, actually doing it is not as easy as it sounds. The students (and teachers!) need to figure out which GeoGebra functionalities to use given a set of construction instructions. That is a good exercise as each step in the construction can then be analyzed by the student.

As a software, GeoGebra has a lot of functionalities. If we are being strict with construction using straight edge and compass, we need to ignore many of the functionalities of GeoGebra. Recently, a game called “Euclid the Game” [14] is becoming popular. The game actually limits the functionalities of GeoGebra, giving a good exercise in construction. Furthermore, this shows that learning construction using GeoGebra can also be fun. As the level of the player in the game progress, more GeoGebra functionalities are being allowed. A similar concept can also be done in classroom discussions for complex construction problems like the ones presented in this paper. Teachers can start with simple and basic construction techniques and when the class progress to the more complicated constructions, they can start using the other GeoGebra functionalities.

The choice of the construction problem used in the classroom discussion is equally important. In this note, we made use of two problems both with great historical background. The historical background can be used as an interesting context at the start of the discussion. Teachers can pose questions such as why the ancient Japanese created the Sangaku problems or how the Archimedean shoemaker problem is a special case of the Problem of Apollonius.

The complexity of the problem is also important as it allows teachers to start at easier construction problems and progress to more difficult and complicated ones, until the main problem is solved. In both examples above, students need to learn how to construct perpendicular lines, how to find the midpoint, how to construct an equilateral triangle, how to find the incenter and construct the incircle. For some students, each of these simple construction problems may be dull when discussed on its own. But when they are discussed in the context of a much more complex problem (such as the examples above), then learning these simple construction problems now has a purpose.

Lastly, the two problems discussed in this paper are just part of a wider collection of problems. The Sangaku problems, while not all are construction problems, consists of many construction problems. A lecture, or series of lectures, can focus on the different Sangaku construction problems. On the other hand, since the Archimedean shoemaker problem is a special case of the Problem of Apollonius, then a lecture can also focus on the complete solution of the Problem of Apollonius.

Students in our college are usually unsurprised if the class will use some sort of software together with their math lessons. Upon using GeoGebra, we did encounter the usual problems students face while using technology in the classroom: the students' unfamiliarity with the software, the learning curve that goes with it, and the usual network problems in a computer laboratory. About half of our students are first generation students and nearing middle age. Using any mathematical software (or any software for that matter) is not natural for them. The teachers will always provide more coaching to these students, and GeoGebra is not an exemption. But what differentiates it from other software is its ability to provide visualization of various mathematical concepts. In the case of the theorems discussed above, students can get convinced of the formulas even before the discussion of the formal and rigorous proof.

During the first semester that GeoGebra was used, one of the authors conducted a simple student survey regarding learning experience using GeoGebra. On a sample of 150 students, all of them rated their experience as either "excellent," "good," or "satisfactory." There was a general appreciation of how the software was able to provide visualization of various math lessons to the students. The usual classroom lecture was still conducted after the laboratory session and this served as a recap of what students learned from the GeoGebra activities.

Looking forward, there are more questions that we need to address. Our college continues to accept the same student demographics, and so we continue to address the issue of first generation, middle aged learners. As to shifting to a more visual software, such as GeoGebra, we need to spend time in designing lessons that make use of such software. Finally, the faculty are having current discussions on how student assessments can be improved, and how GeoGebra can be incorporated in summative assessments.

5. References

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6. Electronic Supplementary Documents

[1] Problem 1 GeoGebra file.
[2] Problem 2 GeoGebra file.
[3] Problem 3 GeoGebra file.